

# Controlled Age-Dependent Population Dynamics Based on Parity Progression

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In this paper, we develop a continuous age-dependent population model based on parity progression, and introduce the age-parity progression fertilities as the control variables of the system. The stationary situation is fully investigated and the distribution of age-parity progression fertilities for the stable system is studied.

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## 1. INTRODUCTION

The “age-dependent parity progression model” is intended to describe the birth process of females by introducing the age distribution density of women who have a given birth order (number of births) and the age-parity-specific fertility. The idea of population dynamics based on parity progression can be traced back to Henry [1], and later Whelpton [2] and Feeney [3]. The advantage of this model as compared to the macroscopic McKendrick population model (see, for example, Song and Yu [4]) is the introduction of the birth control strategy of a given parity instead of that for age-specific fertility and providing of a framework for integrating micro-level birth analysis into macro-level studies of fertility and population growth trends. Furthermore, similar ideas such as the establishing of the parity female ratio model can be used conveniently in population forecasting, see Han [5] for details.

## 2. THE MODEL

We refer, in the following, parity  $n$ ,  $n = 0, 1, 2, \dots, N$ , as the women's birth order,  $N$  is the maximal birth order ever attained by females in a closed

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population. Let  $N_n(r, t)$  be the number of females whose ages are less than  $r$  and with parity  $n$  at time  $t$ . Then the age distribution density of females who have the parity  $n$  is defined as

$$p_n(r, t) = \frac{\partial N_n(r, t)}{\partial r}, \quad n = 0, 1, 2, \dots, N. \quad (1)$$

Let  $\phi_n(r, t) dt$  be the number of women who had  $n-1$  births prior to time  $t$  and have the  $n$ th birth in  $(t, t+dt)$ . The age-parity-specific fertility  $f_n(r, t)$  is defined as the ratio of  $\phi_n(r, t) dt$  to  $p_n(r, t) dt$ , the number of women of parity  $n$ , i.e.,

$$f_n(r, t) = \frac{\phi_n(r, t) dt}{p_n(r, t) dt} = \frac{\phi_n(r, t)}{p_n(r, t)}, \quad n = 1, 2, \dots, N. \quad (2)$$

Obviously,  $f_n(r, t)$  takes zero values outside the fecundity interval  $[r_1, r_2]$ ,  $r_1 > 0$ .

The “age-dependent parity progression model” is a model, based on the  $f_n(r, t)$ , the relative mortality rate  $\mu_f(r, t)$  of females and the births sex ratio  $k_0(t)$ , to describe the evolution process of the age-parity distribution density  $p_n(r, t)$  of females.

These definitions show that in a small enough time interval  $\Delta t$  and age interval  $\Delta r$ , one has

$$\begin{aligned} p_n(r + \Delta t, t + \Delta t) \Delta r \\ = [1 - \mu_f(r, t) \Delta t] [p_n(r, t) \Delta r - p_n(r, t) \Delta r \cdot f_{n+1}(r, t) \Delta t \\ + p_{n-1}(r, t) \Delta r \cdot f_n(r, t) \Delta t], \quad n = 0, 1, \dots, N. \end{aligned} \quad (3)$$

Here and throughout the rest of this paper, we set

$$\begin{aligned} p_n(r, t) &= 0, & \text{for } n < 0 \text{ or } n > N, \\ f_n(r, t) &= 0, & \text{for } n < 1 \text{ or } n > N. \end{aligned}$$

Taking the limit as  $\Delta r \rightarrow 0$ , and  $\Delta t \rightarrow 0$  in (3), one gets the basic system of a partial differential equation

$$\begin{aligned} Dp_n(r, t) &= -\mu_f(r, t) p_n(r, t) - f_{n+1}(r, t) \\ &\quad \cdot p_n(r, t) + f_n(r, t) \cdot p_{n-1}(r, t) \\ &\quad \text{for all } t > 0 \text{ and } 0 < r < r_m, \end{aligned} \quad (4)$$

where  $r_m$  is the age limit of females and

$$Dp_n(r, t) = \lim_{\Delta h \rightarrow 0} \frac{p_n(r + \Delta h, t + \Delta h) - p_n(r, t)}{\Delta h} \quad (5)$$

or

$$Dp_n(r, t) = \frac{\partial p_n(r, t)}{\partial t} + \frac{\partial p_n(r, t)}{\partial r} \quad (6)$$

assuming that  $p_n(r, t)$  is differentiable with respect to  $r$  and  $t$ .

From  $p_n(0, t)$  (the number of female infants entering parity  $n$  at time  $t$ ), one has the following "boundary condition":

$$p_n(0, t) = \begin{cases} 0, & \text{if } n \neq 0, \\ k_0(t) \int_{r_1}^{r_2} \sum_{n=1}^n f_n(r, t) p_{n-1}(r, t) dr, & \text{if } n = 0. \end{cases} \quad (7)$$

Let  $p_n(r, 0) = p_{n0}(r)$  be the initial age distribution of parity  $n$ , we then have the population equation of age-parity progression:

$$\begin{aligned} Dp_n(r, t) &= -\mu_f(r, t) p_n(r, t) - f_{n+1}(r, t) \cdot p_n(r, t) \\ &\quad + f_n(r, t) \cdot p_{n-1}(r, t), \quad t > 0, 0 < r < r_m, \\ p_n(r, 0) &= p_{n0}(r), \quad 0 < r < r_m, \\ p_n(0, t) &= \begin{cases} 0, & \text{if } n \neq 0, \\ k_0(t) \int_{r_1}^{r_2} \sum_{n=1}^N f_n(r, t) p_{n-1}(r, t) dr, & \text{if } n = 0, \end{cases} \end{aligned} \quad (8)$$

where  $k_0(t)$  and  $\mu_f(r, t)$  are the birth sex ratio and relative mortality rate, respectively, and we assume the following conditions are satisfied

$$\int_0^r \mu_f(\rho, t) d\rho < +\infty, \text{ for } r < r_m; \quad \int_0^{r_m} \mu_f(\rho, t) d\rho = +\infty. \quad (9)$$

This model gives finer structures than the McKendrick population model, for example, the  $n$ th fertility  $\beta_n(r, t)$  of females aged  $r$  is given by

$$\beta_n(r, t) = \frac{\phi_n(r, t)}{p_f(r, t)} = \frac{f_n(r, t) p_{n-1}(r, t)}{\sum_{n=0}^N p_n(r, t)}, \quad n = 1, 2, \dots, N, \quad (10)$$

where  $p_f(r, t)$  denotes the age distribution density of females; the average fertility  $f(r, t)$  of all women aged  $r$  and time  $t$

$$f(r, t) = \sum_{n=1}^N \beta_n(r, t); \quad (11)$$

and the age-specific fertility  $\beta(t)$  of females

$$\beta(t) = \int_{r_1}^{r_2} f(r, t) dr. \quad (12)$$

If  $\lambda_n(r, t)$  denotes the female parity ratio,

$$\lambda_n(r, t) = \frac{p_n(r, t)}{p_f(r, t)} = \frac{p_n(r, t)}{\sum_{n=0}^N p_n(r, t)}, \quad n = 0, 1, 2, \dots, N, \quad (13)$$

$$\sum_{n=0}^N \lambda_n(r, t) = 1,$$

then in view of the fact that for small enough  $\Delta r, \Delta t$

$$p_f(r + \Delta t, t + \Delta t) \Delta r = [1 - \mu_f(r, t) \Delta t] p_f(r, t) \Delta r \quad (14)$$

and (3), we obtain immediately the female parity ratio equation

$$\begin{aligned} D\lambda_n(r, t) &= -f_{n+1}(r, t) \cdot \lambda_n(r, t) + f_n(r, t) \cdot \lambda_{n-1}(r, t) \\ \lambda_n(r, 0) &= \lambda_{n0}(r), \\ \lambda_n(0, t) &= \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases} \quad n = 0, 1, \dots, N, \end{aligned} \quad (15)$$

where  $\lambda_{n0}(r)$  is the initial condition, and  $\sum_{n=0}^N \lambda_{n0}(r) = 1$ . In application to population forecasting, Eq. (15) is more convenient since the mortality function disappears in this model. By using  $f_n(r, t)$  and  $\lambda_n(r, t)$ , the  $n$ th fertility  $\beta_n(r, t)$  of females aged  $r$  can be expressed as

$$\beta_n(r, t) = f_n(r, t) \lambda_{n-1}(r, t), \quad n = 1, 2, \dots, N. \quad (16)$$

Furthermore, since the fertility pattern  $h(r, t)$  (see [4]) is given by

$$h(r, t) = \frac{f(r, t)}{\beta(t)}$$

we can show that the McKendrick type female population equation becomes

$$\begin{aligned} Dp_f(r, t) &= -\mu_f(r, t) p_f(r, t), \\ p_f(r, 0) &= p_{f0}(r), \\ p_f(0, t) &= k_0(t) \int_{r_1}^{r_2} \sum_{n=1}^N f_n(r, t) \lambda_{n-1}(r, t) p_f(r, t) dt. \end{aligned} \quad (17)$$

Hence by  $p_f(r, t)$ , the system (8) and (15) can be deduced from each other. However, the former is linear and hence its mathematical treatment will be more convenient.

From Eq. (8), we see that when the initial conditions  $p_{n0}(r)$ ,  $n = 0, 1, \dots, N$  are given, the age-parity density  $p_n(r, t)$  can be determined by the age-parity-specific fertility  $f_n(r, t)$  and mortality  $\mu_f(r, t)$ . We may thus consider the  $f_n(r, t)$ ,  $n = 1, 2, \dots, N$ , as the control variables of the age-parity

progression system. The other parameter  $\mu_f(r, t)$  is not a control variable in general. A birth control or family-planning program is essentially the regulation of  $f_n(r, t)$ . For example, in the one-child program one just sets  $f_2 = f_3 = \dots = f_N = 0$ . A two-child program is equivalent to the case  $f_3 = f_4 = \dots = f_N = 0$ , etc. On the other hand, in order to establish the population control model corresponding to various birth policies we must know what is the "standard age-parity progression fertilities" or how to embody the effect of different birth policies on  $f_n$ . The "standard age-parity progression fertilities" here refers to the average age-parity progression fertilities of women under the natural state (i.e., no artificial means are imposed). For example, about 2% of women are sterile, a little above 2% of women have only one child, etc. The standard age-parity progression fertilities are denoted by

$$h_n(r, t), \quad n = 1, 2, \dots, N. \quad (18)$$

In a stationary situation we can assume  $h_n(r, t) = h_n(r)$ , independent of time  $t$ . The actual programmed age-parity progression fertilities are hence expressed as

$$f_n(r, t) = \beta_n(t) h_n(r, t), \quad n = 1, 2, \dots, N. \quad (19)$$

The parameters  $\beta_n(t)$ ,  $0 \leq \beta_n(t) \leq 1$ , appearing in (19) indicate clearly the relative level of the actual age-parity progression fertilities compared to the standard age-parity progression fertilities. Hence, they constitute a group of control variables corresponding to various birth policies. For example, when only "one child" is allowed for every fertile woman, this is equivalent to  $\beta_1 = 1$ , and  $\beta_2 = \beta_3 = \dots = \beta_N = 0$ ; when only "two children" are permitted for every fertile women is equivalent to  $\beta_1 = \beta_2 = 1$ ,  $\beta_3 = \beta_4 = \dots = \beta_N = 0$ ; when "one child" is allowed for all fertile women but only 20% of women who have had one child previously are allowed to have a second child and more than two children are not permitted is equivalent to  $\beta_1 = 1$ ,  $\beta_2 = 0.2$ ,  $\beta_3 = \beta_4 = \dots = \beta_N = 0$ .

### 3. THE STATIONARY CASE

In this section, we will consider the stationary age-parity progression system, i.e.,  $\beta_n(t) = \beta_n$ ,  $k_0(t) = k_0$ ,  $\mu_f(r, t) = \mu_f(r)$ ,  $f_n(r, t) = \beta_n h_n(r)$  are independent of time  $t$  for all  $n$ ,  $n = 1, 2, \dots, N$ , and measurable respect to their variables in Eq. (8). This system involves nonlocal boundary conditions and describes, to a certain extent, the birth dynamical process of a stationary closed population within a short period of time. We shall show that a semigroup can be associated to it and identify the infinitesimal generator. Its spectral properties are analyzed yielding large time behaviour.

Now, the system is

$$\begin{aligned} \frac{\partial p_n(r, t)}{\partial t} + \frac{\partial p_n(r, t)}{\partial r} &= -\mu_f(r) p_n(r, t) - \beta_{n+1} h_{n+1}(r) \cdot p_n(r, t) \\ &\quad + \beta_n h_n(r) \cdot p_{n-1}(r, t), \quad t > 0, 0 < r < r_m, \\ p_n(r, 0) &= p_{n0}(r), \quad 0 < r < r_m, \\ p(0, t) &= \begin{cases} 0, & \text{if } n \neq 0, \\ k_0 \int_{r_1}^{r_2} \sum_{n=1}^N \beta_n h_n(r) p_{n-1}(r, t) dr, & \text{if } n = 0, \end{cases} \end{aligned} \quad (20)$$

where  $0 \leq \beta_n, h_n(r) \leq 1$ ,  $\text{mes}\{r \in [r_1, r_2] \mid h_n(r) \neq 0\} > 0$  for  $n = 1, 2, \dots, N$ .

For  $n = 1, 2, \dots, N$ , let

$$\begin{aligned} \mathbf{P}(r, t) &= \begin{Bmatrix} p_0(r, t) \\ p_1(r, t) \\ \vdots \\ p_N(r, t) \end{Bmatrix}, \quad \mathbf{P}_0(r) = \begin{Bmatrix} p_{00}(r) \\ p_1(r) \\ \vdots \\ p_N(r) \end{Bmatrix} \\ \mathbf{A}_n(r) &= \begin{Bmatrix} 0 & 0 & \cdots & 0 & \cdots \cdots \cdots & 0 \\ 0 & 0 & & & & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & h_n(r) & & & \vdots \\ \vdots & & -h_n(r) & & & \vdots \\ 0 & 0 & \cdots \cdots \cdots & 0 & \cdots \cdots \cdots & 0 \end{Bmatrix}_{(N+1) \times (N+1)} \begin{matrix} (n, n) \\ (n+1, n) \end{matrix} \\ \mathbf{B}_n(r) &= k_0 \begin{Bmatrix} 0 & 0 & \cdots & h_n(r) & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{Bmatrix}_{(N+1) \times (N+1)} \end{aligned} \quad (21)$$

then Eq. (20) can be written as

$$\begin{aligned} \frac{\partial \mathbf{P}(r, t)}{\partial t} + \frac{\partial \mathbf{P}(r, t)}{\partial r} &= \left[ -\mu_f(r) - \sum_{n=1}^N \beta_n \mathbf{A}_n(r) \right] \mathbf{P}_n(r, t), \\ \mathbf{P}(r, 0) &= \mathbf{P}_0(r), \\ \mathbf{P}(0, t) &= \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{P}(r, t) dr. \end{aligned} \quad (22)$$

Let  $T(r_0, r)$  be the evolution matrix of the ordinary differential equation

$$\frac{dg(r)}{dr} = \left[ -\mu_f(r) - \sum_{n=1}^N \beta_n \mathbf{A}_n(r) \right] g(r).$$

Define  $\mathbf{Q}(r, t)$  as the solution of

$$\mathbf{Q}(r, t) = \begin{cases} \mathbf{T}(r-t, t) \mathbf{P}_0(r-t), & r \geq t, \\ \mathbf{T}(0, r) \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{Q}(s, t-r) ds, & r < t, \end{cases} \quad (23)$$

then arguments similar to those used in Guo and Chan [6] lead us to

**PROPOSITION 1.** *For any  $\mathbf{P}_0(r) \in \mathbf{X} = (L^p(0, r_m))^{N+1}$ , there exists a unique solution  $\mathbf{Q}(r, t)$  to (23),  $\mathbf{Q}(r, t) \in \mathbf{X}$ ,  $1 \leq p < \infty$ . The operators  $\mathbb{T}(t): \mathbf{X} \rightarrow \mathbf{X}$ , for all  $t \geq 0$ , defined by*

$$\mathbb{T}(t) \mathbf{P}_0(r) = \mathbf{Q}(r, t) \quad (24)$$

*form a strongly continuous semigroup of  $\mathbf{X}$ , with the infinitesimal generator  $\mathbb{A}$ :*

$$\begin{aligned} D(\mathbb{A}) &= \left\{ \Phi(r) \mid \Phi, \mathbb{A}\Phi \in \mathbf{X}, \Phi(0) = \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \Phi(r) dr \right\} \\ \mathbb{A}\Phi(r) &= -\Phi'(r) - \left[ \mu_f(r) + \sum_{n=1}^N \beta_n \mathbf{A}_n(r) \right] \Phi(r). \end{aligned} \quad (25)$$

With operator  $\mathbb{A}$ , we can write Eq. (22) as an abstract evolution equation in  $\mathbf{X}$

$$\begin{aligned} \frac{d\mathbf{P}(r, t)}{dt} &= \mathbb{A}\mathbf{P}(r, t), \\ \mathbf{P}(r, 0) &= \mathbf{P}_0(r). \end{aligned} \quad (26)$$

By Proposition 1, we immediately have

**THEOREM 1.** *There exists a unique solution  $\mathbf{P}(r, t)$  to Eq. (26) and*

(i)  $\mathbf{P}(r, t) = \mathbb{T}(t) \mathbf{P}_0(r) \in C([0, \infty); \mathbf{X})$ ,  $\forall \mathbf{P}_0 \in \mathbf{X}$ ,  $\mathbf{P}(r, t) = \mathbb{T}(t) \mathbf{P}_0(r) \in C^1([0, \infty); \mathbf{X})$ ,  $\forall \mathbf{P}_0 \in D(\mathbb{A})$ ;

(ii)  $\mathbb{T}(t)$  is compact in  $\mathbf{X}$  for  $t \geq r_m$ , but is not for  $t < r_m$  and hence does not have an analytic extension. Furthermore,

$$\mathbb{T}(t) \mathbf{P}_0(r) = \begin{cases} \mathbf{T}(r-t, t) \mathbf{P}_0(r-t), & r \geq t, \\ \mathbf{T}(0, r) \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{T}(s+r-t, t-r) \mathbf{P}_0(s+r-t) ds, & r < t \end{cases} \quad \text{for } t \leq r_1,$$

$$\mathbb{T}(t) = [\mathbb{T}(r_1)]^{\lceil t/r_1 \rceil} \mathbb{T} \left( t - \left\lfloor \frac{t}{r_1} \right\rfloor r_1 \right), \quad \text{for } t > r_1. \quad (27)$$

Next, we shall investigate the spectrum of the operator  $\mathbb{A}$ . Take  $\lambda \in \mathbb{C}$ ,  $\Psi(r) \in \mathbb{X}$ , solve the equation

$$(\lambda - \mathbb{A}) \Phi(r) = \Psi(r), \quad (28)$$

i.e.,

$$\frac{d\Phi(r)}{dr} = \left[ -\lambda - \mu_f(r) - \sum_{n=1}^N \beta_n \mathbf{A}_n(r) \right] \Phi(r) + \Psi(r),$$

$$\Phi(0) = \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \Phi(r) dr,$$

and then one has

$$\Phi(r) = \mathbf{T}(0, r) \Phi(0) e^{-\lambda r} + \int_0^r \mathbf{T}(0, r-s) \Psi(s) e^{-\lambda(r-s)} ds. \quad (29)$$

If

$$\det \left( I - \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{T}(0, r) e^{-\lambda r} dr \right) \neq 0 \quad (30)$$

then  $\lambda \in \rho(\mathbb{A})$ , the resolvent set of  $\mathbb{A}$ , and

$$R(\lambda, \mathbb{A}) \Psi(r) = \mathbf{T}(0, r) \left( I - \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{T}(0, r) e^{-\lambda r} dr \right)^{-1}$$

$$\times \int_{r_1}^{r_2} \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \cdot \int_0^r \mathbf{T}(0, r-s) \Psi(s) e^{-\lambda(r-s)} ds dr$$

$$\cdot e^{-\lambda r} + \int_0^r \mathbf{T}(0, r-s) \Psi(s) e^{-\lambda(r-s)} ds. \quad (31)$$

If

$$\det \left( I - \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{T}(0, r) e^{-\lambda r} dr \right) = 0 \quad (32)$$



then  $\hat{\lambda} \in \sigma_p(\mathbb{A})$ , the point spectrum of  $\mathbb{A}$ , and the corresponding eigenfunction is

$$\Phi(r) = \mathbf{T}(0, r) \Phi_0 \cdot e^{-\hat{\lambda} r}, \quad (33)$$

where  $\Phi_0$  is a  $(N+1) \times 1$  vector satisfying

$$\left( I - \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \mathbf{T}(0, r) e^{-\hat{\lambda} r} dr \right) \Phi_0 = 0. \quad (34)$$

On the other hand, it can be verified directly that (30) is equivalent to

$$F(\lambda) = 1 - k_0 \int_{r_1}^{r_2} \sum_{n=1}^N \beta_n h_n(r) \hat{\phi}_{n-1} e^{-\lambda r - \int_0^r \mu_f(\rho) d\rho} dr \neq 0, \quad (35)$$

where

$$\begin{aligned} \hat{\phi}_0(r) &= e^{-\beta_1 \int_0^r h_n(\rho) d\rho}, \\ \hat{\phi}_n(r) &= \int_0^r \beta_n h_n(s) e^{-\beta_{n-1} + 1 \int_0^s h_{n+1}(\rho) d\rho} \hat{\phi}_{n-1}(s) ds, \quad n = 1, 2, \dots, N, \end{aligned} \quad (36)$$

and when  $F(\lambda) = 0$ , there is only one linear independent solution to (34), and the corresponding eigenfunction is of the form

$$\begin{aligned} \Phi(r) &= c \hat{\Phi}(r) e^{-\lambda r - \int_0^r \mu_f(\rho) d\rho}, \\ \hat{\Phi}(r) &= (\hat{\phi}_n(r))_{(N+1) \times 1}, \quad c \neq 0. \end{aligned} \quad (37)$$

Consequently, we can prove results analogous to those of the McKendrick type population equation [4], i.e.,

$$\sigma(\mathbb{A}) = \sigma_p(\mathbb{A}) = \{\hat{\lambda} \in \mathbb{C} \mid F(\hat{\lambda}) = 0\}, \quad (38)$$

and

**THEOREM 2.** (i) *The point spectrum of  $\mathbb{A}$  consists of distinct eigenvalues of geometric multiplicity 1. They consist of the zeros of the entire function  $F(\lambda)$ .*

(ii)  *$\mathbb{A}$  has only one real eigenvalue  $\lambda_0$ , its algebraic multiplicity is 1.*

(iii) *There is only a finite number of eigenvalues of  $\mathbb{A}$  in any finite strip parallel to the imaginary axis.*

(iv)  *$\sigma(\mathbb{A})$  is an infinite set.*

(v) *The solution  $\mathbf{P}(r, t)$  of Eq. (22) has the asymptotic expansion*

$$\mathbf{P}(r, t) = \mathbb{P}_{\lambda_0} \mathbf{P}_0(r) \cdot e^{\lambda_0 t} + o(e^{(\lambda_0 - \varepsilon)t}), \quad \text{as } t \rightarrow \infty, \quad (39)$$

where  $\varepsilon > 0$  is a small number such that  $\sigma(\mathbb{A}) \cap \{\lambda \mid \lambda_0 - \varepsilon < \operatorname{Re} \lambda < \lambda_0\} = \emptyset$ ,  $\lambda_0$  is the dominant real eigenvalue called the growth index and

$$\mathbb{P}_{\lambda_0} \mathbf{P}_0(r) = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0) R(\lambda, \mathbb{A}) \mathbf{P}_0(r). \quad (40)$$

(vi) Let  $N_n(t) = \int_0^t p_n(r, t) dr$  be the number of women who have parity  $n$  at time  $t$ , then when  $\lambda_0 = 0$ , there is a constant  $N_n^*$  such that

$$\lim_{t \rightarrow \infty} N_n(t) = N_n^*.$$

The convergence is in a damped oscillatory fashion.

When  $\lambda_0 = 0$ , Theorem 2 tells us that

$$\lim_{t \rightarrow \infty} \mathbf{P}(r, t) = \mathbb{P}_0 \mathbf{P}_0(r).$$

$\Phi_0(r) = \mathbb{P}_0 \mathbf{P}_0(r)$  is the nonnegative equilibrium state of system (22), i.e.,

$$\begin{aligned} \frac{d\Phi_0(r)}{dr} &= \left[ -\mu_f(r) - \sum_{n=1}^N \beta_n \mathbf{A}_n(r) \right] \Phi_0(r) \\ \Phi_0(0) &= \int_{r_1}^{r_2} \left[ \sum_{n=1}^N \beta_n \mathbf{B}_n(r) \right] \Phi_0(r) dr. \end{aligned}$$

For the female parity ratio equation (15), similar conclusions are apparent. Assume also that  $f_n(r, t) = \beta_n h_n(r)$ , and the solution of (15) exists (e.g., under some smooth conditions on the initial function  $\lambda_{n0}(r)$ ), then integrating along the characteristics, one has

$$\begin{aligned} \lambda_0(r, t) &= \begin{cases} \lambda_0(r-t) e^{-\beta_1 \int_{r-t}^r h_1(\rho) d\rho}, & r \geq t, \\ e^{-\beta_1 \int_0^r h_1(\rho) d\rho}, & r < t \end{cases} \\ \lambda_n(r, t) &= \lambda_{00}(r-t) e^{-\beta_{n+1} \int_{r-t}^r h_{n+1}(\rho) d\rho} \\ &\quad + \int_{t-r}^t \beta_n h_n(r-t+s) \lambda_{n+1}(r-t+s, s) \\ &\quad \cdot e^{-\beta_{n+1} \int_{r-t+s}^r h_{n+1}(\rho) d\rho} ds, \quad n = 1, 2, \dots, N. \end{aligned} \quad (41)$$

A straightforward calculation shows that

$$\lambda_n(r, t) = \lambda_n(r), \quad \text{for all } t > r \text{ and } n = 1, 2, \dots, N. \quad (42)$$

$\lambda_n(r)$  is the nonzero equilibrium state of system (15)

$$\begin{aligned} \frac{d\lambda_n(r)}{dr} &= -\beta_{n+1}h_{n+1}(r) \cdot \lambda_n(r) + \beta_n h_n(r, t) \cdot \lambda_{n-1}(r) \\ \lambda_n(0, t) &= \begin{cases} 1, & \text{if } n=0, \\ 0, & \text{if } n \neq 0, \end{cases} \quad n=0, 1, \dots, N, \end{aligned} \quad (43)$$

i.e.,

$$\lambda_n(r) = \hat{\phi}_n(r), \quad n=0, 1, \dots, N, \quad (44)$$

where  $\hat{\phi}_n(r)$  is defined in (36).

*Note.* We want to point out a very interesting fact. Let  $F(\lambda)$  be defined as in (35), then a calculation shows that

$$\begin{aligned} F(0) &= 1 - Nk_0 + Nk_0 \int_0^{r_m} \mu_f(r) e^{-\int_0^r \mu_f(\rho) d\rho} e^{-\beta_1 \int_0^r h_1(\rho) d\rho} dr \\ &\quad + k_0 \sum_{n=2}^N \sum_{m=1}^{n-1} \int_0^{r_m} \mu_f(r) e^{-\int_0^r \mu_f(\rho) d\rho} \\ &\quad \cdot \int_0^r \beta_m h_m(s) \hat{\phi}_{m-1}(s) e^{-\beta_{m+1} \int_s^r h_{m+1}(\rho) d\rho} ds. \end{aligned} \quad (45)$$

If  $k_0 = 1/N$  (e.g.,  $N=2$ ,  $k_0 = 1/2$ ), then from (45),  $F(0) > 0$ , so the growth index  $\lambda_0 < 0$ , and hence

$$\lim_{t \rightarrow \infty} \mathbf{P}(r, t) = 0.$$

Finally, we are in the position to discuss the distribution of  $(\beta_1, \beta_2, \dots, \beta_N)$  leading to the growth index  $\lambda_0 = 0$ . This seems very important in practice since the final target of the control of the population is to reduce the growth index to zero.

**THEOREM 3.** *If under the natural state (i.e.,  $\beta_1 = \beta_2 = \dots = \beta_N = 1$ ), the growth index is greater than zero, then there exists a  $(N-1)$  dimension smooth surface  $S^{N-1}$  connected with axis*

$$S^{N-1}: \beta_N = g(\beta_1, \beta_2, \dots, \beta_{N-1}) \quad (46)$$

such that (see Fig. 1)

(i) when  $(\beta_1, \beta_2, \dots, \beta_N)$  lies above  $S^{N-1}$ , the corresponding growth index is  $\lambda_0 > 0$ ;

(ii) when  $(\beta_1, \beta_2, \dots, \beta_N)$  lies on  $S^{N-1}$ ,  $\lambda_0 = 0$ ;

(iii) when  $(\beta_1, \beta_2, \dots, \beta_N)$  lies below  $S^{N-1}$ ,  $\lambda_0 < 0$ .

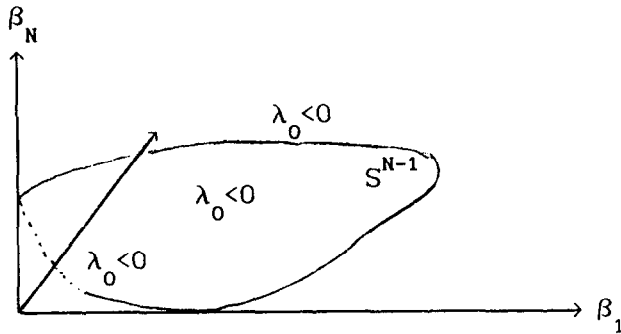


FIG. 1. The distribution of age-parity fertilities of stable populations.

*Proof.* From previous discussion,  $\lambda_0 = 0$  if and only if  $F(0) = 0$ . Let

$$G(\beta_1, \beta_2, \dots, \beta_N) = F(0), \quad \text{for all } (\beta_1, \beta_2, \dots, \beta_N) \in \Omega_N, \quad (47)$$

where  $\Omega_N = \{(\beta_1, \beta_2, \dots, \beta_N) \mid 0 \leq \beta_n \leq 1, n = 1, 2, \dots, N\}$ . By (45),

$$\begin{aligned} \frac{\partial G(\beta_1, \beta_2, \dots, \beta_N)}{\partial \beta_N} &= -k_0 \int_0^{r_m} \mu_f(r) e^{-\int_0^r \mu_f(\rho) d\rho} \int_0^r h_N(\rho) d\rho \\ &\quad \cdot \int_0^\rho \beta_{N-1} h_{N-1}(s) \hat{\phi}_{N-2}(s) e^{-\beta_N \int_s^r h_N(\rho) d\rho} ds dr \neq 0 \\ &\quad \text{for all } (\beta_1, \beta_2, \dots, \beta_N) \in \dot{\Omega}_N. \end{aligned} \quad (48)$$

If  $G(\beta_1, \beta_2, \dots, \beta_N) = F(0) = 0$ , for some  $(\beta_1, \beta_2, \dots, \beta_N) \in \dot{\Omega}_N$ , then by the implicit function theorem, there exists an open neighborhood of  $(\beta_1, \beta_2, \dots, \beta_N)$  such that

$$\beta_N = g(\beta_1, \beta_2, \dots, \beta_{N-1}). \quad (49)$$

On the other hand, if  $(\beta_1, \beta_2, \dots, \beta_N) \geq (\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_N)$ , i.e.,  $\beta_n \geq \bar{\beta}_n$  for all  $n = 1, 2, \dots, N$ , then through the following programs

$$\begin{aligned} (\beta_1, \beta_2, \dots, \beta_{N-1}, \beta_N) &\rightarrow \lambda_0 \\ (\beta_1, \beta_2, \dots, \beta_{N-1}, \bar{\beta}_N) &\rightarrow \lambda_{0N} \\ &\dots\dots\dots \\ (\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_{N-1}, \bar{\beta}_N) &\rightarrow \bar{\lambda}_0 \end{aligned}$$

the right hand sides are the growth index's corresponding to vectors on the left hand sides. We can prove from the expression of  $F(0)$  that

$$\lambda_0 \geq \lambda_{0N} \geq \dots \geq \bar{\lambda}_0.$$

So  $\lambda_0 \geq \bar{\lambda}_0$ . This demonstrates that the growth index  $\lambda_0$  is a monotonic function of  $(\beta_1, \beta_2, \dots, \beta_N)$  with respect to the nonnegative cone of  $\mathbb{R}^N$ . Furthermore, from

$$e^{j_0 t} = \lim_{n \rightarrow \infty} (\|\mathbb{T}(t)^n\|^{1/n}),$$

and (27),  $\lambda_0$  is also continuous with respect to  $(\beta_1, \beta_2, \dots, \beta_N)$ .

Let  $(\beta_1, \beta_2, \dots, \beta_N) \in \Omega_N$ , in which  $\beta_n$ ,  $n = 1, 2, \dots, N$ , are small enough such that

$$G(\beta_1, \beta_2, \dots, \beta_N) > 0$$

then the corresponding growth index  $\lambda_{00} < 0$ . For  $(\beta_1, \beta_2, \dots, \beta_N) \in \Omega_N$  in which  $\beta_n = 1$ ,  $n = 1, 2, \dots, N$ , there are two cases:

*Case 1.* The corresponding growth index  $\lambda_{10} \leq 0$  (e.g., the death rate is very large);

*Case 2.*  $\lambda_{10} > 0$ .

In Case 1, the distribution of  $(\beta_1, \beta_2, \dots, \beta_N)$  corresponding to the zero growth may lie on some branches of  $(N-1)$  dimension surfaces. Our assumption guarantees that Case 1 is impossible. Usually, Case 2 is most probable. In such a case, by the continuous dependence of  $\lambda_0$  with respect to  $(\beta_1, \beta_2, \dots, \beta_N)$ , all  $(\beta_1, \beta_2, \dots, \beta_N)$  corresponding to the zero growth index forms a  $(N-1)$  dimension smooth surface, connected with the coordinate axis. That surface can be represented by

$$\beta_N = g(\beta_1, \beta_2, \dots, \beta_{N-1}) \quad \text{for all } (\beta_1, \beta_2, \dots, \beta_{N-1}) \in \Omega_{N-1}, \quad (50)$$

where  $\Omega_{N-1} = \{(\beta_1, \beta_2, \dots, \beta_{N-1}) | 0 \leq \beta_n \leq 1, n = 1, 2, \dots, N-1\}$ . The other conclusions are followed from the continuous dependence of  $\lambda_0$  with respect to  $(\beta_1, \beta_2, \dots, \beta_N)$ .

From (10), the age-specific fertility of the McKendrick type equation is

$$\beta(t) = \int_{r_1}^{r_2} \sum_{n=1}^N \beta_n h_n(r) \lambda_{n-1}(r, t) dr \quad (51)$$

when  $t > r_2$ , from (42) we see that

$$\beta(t) = \beta = \int_{r_1}^{r_2} \sum_{n=1}^N \beta_n h_n(r) \lambda_{n-1}(r) dr. \quad (52)$$

The critical age-specific fertility  $\beta_{cr}$  is defined as the solution of (see [4])

$$1 - k_0 \beta_{cr} \int_{r_1}^{r_2} h(r) h(r) e^{-\int_0^r \mu_f(\rho) d\rho} d\rho = 0. \quad (53)$$

Comparing the two types of equations, it is not difficult to deduce that the growth index of them are the same, and

$$\beta = \beta_{cr} \quad \text{if and only if } (\beta_1, \beta_2, \dots, \beta_N) \text{ lies on } S^{N-1}. \quad (54)$$

For this reason we can say that  $S^{N-1}$  is the critical age-parity progression fertility surface.

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